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Investigating a Large Amplitude Pendulum

Research Question

How does the amplitude of a pendulum's oscillation affect its period?

Introduction

I have been fascinated by pendulums for a long time, ever since I first saw one in a museum. I recently watched an online lecture in which a professor sits on a large pendulum to show that the period is not affected by mass. This sparked my interest, and made me wonder about the various factors that did or didn't affect the period. I decided to examine the amplitude and period relationship.

We learned in class that pendulum oscillations were an example of simple harmonic motion. Hence, they were presumed to be isochronous, meaning that the period wasn't affected by the amplitude. However, this seemed counterintuitive and wrong to me. Additionally, the large number of assumptions made during the equation derivation restricted the applicability of the model to small angle oscillations. I therefore decided to come up with a more accurate model of the pendulum oscillation's period, without using the small-angle approximation. I will be comparing the results of the classical model, the new large amplitude model, and an experimental pendulum.

Modelling

Typically, when coming up with an expression for the pendulum's period, the small angle approximation of $\sin\theta \approx \theta$ is used. This yields the familiar equation:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the pendulum's string length, and g the effective acceleration due to gravity.

When coming up with the more accurate model, we will still need to make certain assumptions. These are that there is no friction at the top joint of the pendulum, that the string is massless and rigid, and that the bob is miniscule.

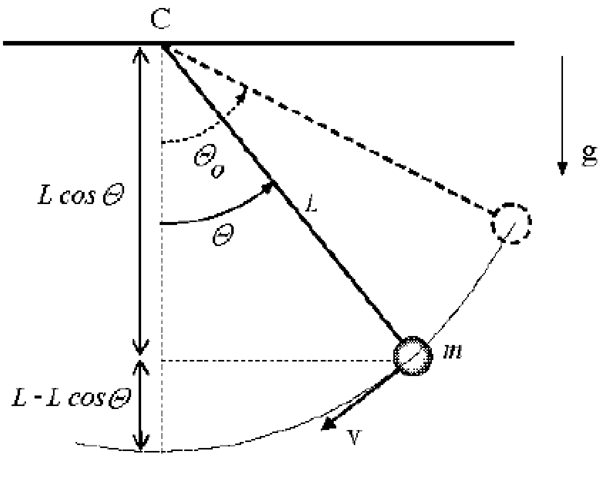


Figure 1, Pendulum (Lima and Arun)

We can use the principle of the conservation of energy to derive the equation for the pendulum's motion, and then the period. The energy in this pendulum is always either in the form of kinetic or gravitational potential energy. The gravitational potential energy depends on the bob's height relative to the equilibrium point, which can be shown as $L - Lcos\theta = L(1 - cos\theta)$. The gravitational potential energy at any point is therefore $mgL(1 - cos\theta)$. When θ_0, which is the angular amplitude, is plugged into the equation, this will give us the total energy of the system.

Next, we need the kinetic energy of the bob. We know that $s = \theta * L$ from circular motion, therefore $v = \frac{d\theta}{dt} * L$. The kinetic energy becomes $\frac{1}{2}mv^2 = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2$.

The following equality for the energy of the system can then be formed:

$$mgL(1 - cos\theta_0) = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mgL(1 - cos\theta)$$

When written in terms of dθ/dt:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}(cos\theta - cos\theta_0)}$$

Next, the equation is written in terms of dt, so that we can get an expression for the period:

$$dt = \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta$$

And finally, we integrate both sides. The right-hand-side is integrated from 0 to θ_0 and then multiplied by 4, because the pendulum travels from the equilibrium position to the extreme position four times in a complete oscillation. This yields the expression for the period:

$$T = 4\sqrt{\frac{l}{2g}} * \int_0^{\theta_0} \frac{1}{\sqrt{cos\theta - cos\theta_0}}d\theta$$

This integral cannot be solved analytically, and must instead be evaluated numerically (Lima and Arun). In this investigation, this will be computed with the Wolfram Alpha platform, but any capable calculator or computer software can also be used.

Hypothesis

My hypothesis is that the expression above will be more accurate than the traditional pendulum equation, especially when the amplitude of the oscillation is relatively large, because it doesn't rely on the small-angle approximation. We will be comparing both models with the experimental results to see how they differ.

Variables

Independent variable:

The amplitude of the oscillation. This will be equal to the angle between the maximum position of the pendulum and the vertical equilibrium position, and will be expressed in degrees.

The variable will be varied from 10 degrees to 90 degrees, with 10 degree increments. Each case will be repeated 5 times to minimize the random error.

Instead of measuring the angle directly, the independent variable will be measured and changed in terms of the horizontal distance of the pendulum bob from the equilibrium position, for greater ease and precision. As the length of the string will be 1 meter, this horizontal distance will be equal to $1*\sin(\theta)$. The horizontal distances will therefore be 0.174, 0.342, 0.500, 0.643, 0.766, 0.866, 0.940, 0.985 and 1.00 meters.

Dependent variable:

The period of the pendulum's oscillation. This will be measured with a handheld stopwatch. After the pendulum is pulled to the side and let go, it will be allowed to oscillate once and come back to its initial maximum position, at which point the stopwatch will be started; this is to make sure that the operator's error is minimal, and the pendulum is oscillating regularly. The pendulum will then be allowed the oscillate 10 times, before the stopwatch is stopped. The recorded time will be divided by 10, to achieve a precise result for the period.

Control variables:

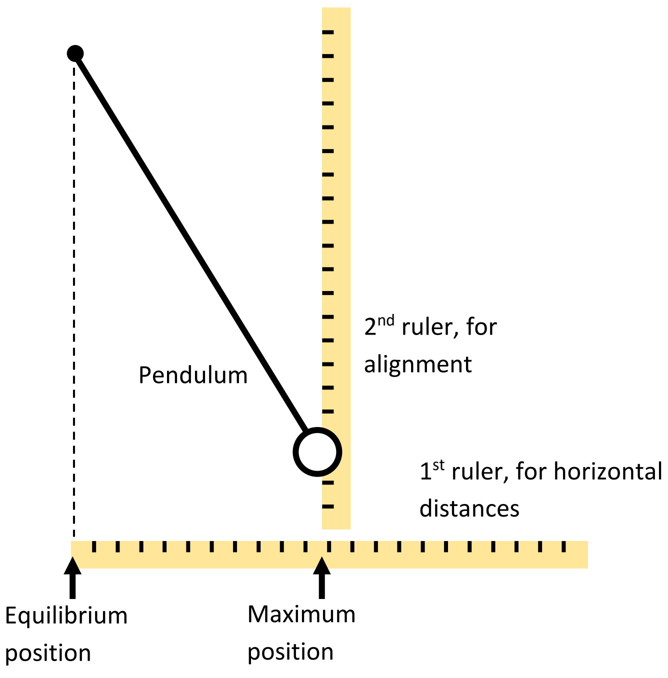
- Length and material of the pendulum's string:
  - Silk thread of length 1 meter, as measured by a meter-ruler.
  - Reasons:
    - Length affects the period, as shown by the pendulum formula.
    - Silk thread is very light, which fits the assumption of it being massless.
    - The thread is also thin, thus minimizing the drag force.
- Mass and material of the pendulum's bob:
  - 65.0 ±0.5 gram lead bob, with a hook to tie the string to.
  - Reason:
    - Lead is a dense metal, which will maximize mass while minimizing the surface area, thereby minimizing the drag force.
- Initial velocity of pendulum:
  - When the pendulum is released from its maximum position, it will be let go without pushing so that its initial velocity is close to zero.
  - Reason:

    - If the velocity isn't zero at the measured maximum position, then its amplitude will actually be greater than expected, making the measurements inaccurate.
- Medium that the pendulum oscillates in:
  - The pendulum will be oscillating in air.
  - Reason:
    - Placing the pendulum in a different medium, such as water, will not only affect the drag force, but there will also be a buoyancy force changing the effective acceleration and hence period.

Materials List

- 1-meter silk thread
- 2 meter-rulers
- 65g lead bob
- Stopwatch

Diagram



Method

1. Tie one end of the silk thread to a point on the ceiling.
2. Tie the lead bob to the bottom end of the fishing rod, so that the string length between the top and the center of mass of the bob is 1 meter.
3. Cut the excess fishing rod from the bottom.
4. Place one of the meter-rulers under the pendulum horizontally, with its zero-mark aligned with the pendulum's equilibrium position.
5. Hold the other meter-ruler perpendicular to the first, at the 0.174 meter position of the horizontal ruler. Move the bob to align with the vertical ruler.

6. Let go of the bob, and wait for it to complete its first oscillation.
7. When the first oscillation is complete, and the bob is momentarily at its initial position again, start the timer on the stopwatch.
8. Wait for the pendulum to make 10 full oscillations. Stop the timer when it finishes.
9. Divide the time measured by 10 to get the period. Record this in the table below.
10. Repeat steps 5-9, four more times (to get a total of 5 trials).
11. Repeat steps 5-11 with each of the other amplitudes.

Results

Table 1 – Amplitudes and Experimental Time Periods of Pendulum

	Time for 10 oscillations / sec ΔT = ±0.4 s						
Amplitude / degrees Δθ = ±5°	**Trial 1**	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Mean**	**Experimental Period / sec ΔT = ±0.04 s**
10	20.19	20.21	20.31	20.31	20.18	20.24	2.02
20	20.27	20.39	20.39	20.47	20.17	20.34	2.03
30	20.46	20.51	20.55	20.51	20.45	20.50	2.05
40	20.56	20.74	20.86	20.65	20.61	20.68	2.07
50	21.08	20.92	21.06	21.03	21.15	21.05	2.10
60	21.21	21.29	21.52	21.34	21.43	21.36	2.14
70	21.5	21.56	21.68	21.72	21.76	21.64	2.16
80	21.97	22.15	22.13	22.17	22.06	22.10	2.21
90	22.42	22.45	22.56	22.63	22.55	22.52	2.25

Observations:

When letting go of the pendulum at large angles, the string wouldn't be taut for the first few oscillations.

The oscillations were dampened significantly (almost by half after the 10 oscillations). This was most noticeable in the large angled oscillations.

The pendulum wasn't just moving back and forth; it was also drifting off to the sides, performing circular motion.

Uncertainties

The most significant uncertainty in the time measurements is the human reaction time, rather than the stopwatch's resolution. The human reaction time is found to be 0.2 seconds on average (Kosinski). As this error is introduced during both the initiation and ending of the time measurement, the error was taken as ±0.4 seconds. As each time measurement consisted of 10 oscillations, the uncertainty of the periods was then taken as 0.4 / 10 = 0.04 seconds.

The standard deviations of the time measurements were also evaluated, using the following expression:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

However, all of the standard deviations were found to be under 0.1, which is smaller than the value of 0.4 found above, which is why the standard deviation wasn't used as the uncertainty.

There was also a relatively large amount of uncertainty introduced when measuring the amplitudes using the horizontal distance method. A reasonable value for this uncertainty is ±5 degrees, as this is equal to half of the independent variable's increment value (10 degrees) and should be sufficient to account for the random errors.

Analysis

The simple pendulum model was used to calculate the isochronous period as:

$$T = 2\pi\sqrt{\frac{1}{9.81}} = 2.006066681\ s$$

Next, the new large amplitude model was used to predict the periods for each value of the amplitude. For example, to find the period for 10 degrees, the following expression was written and evaluated:

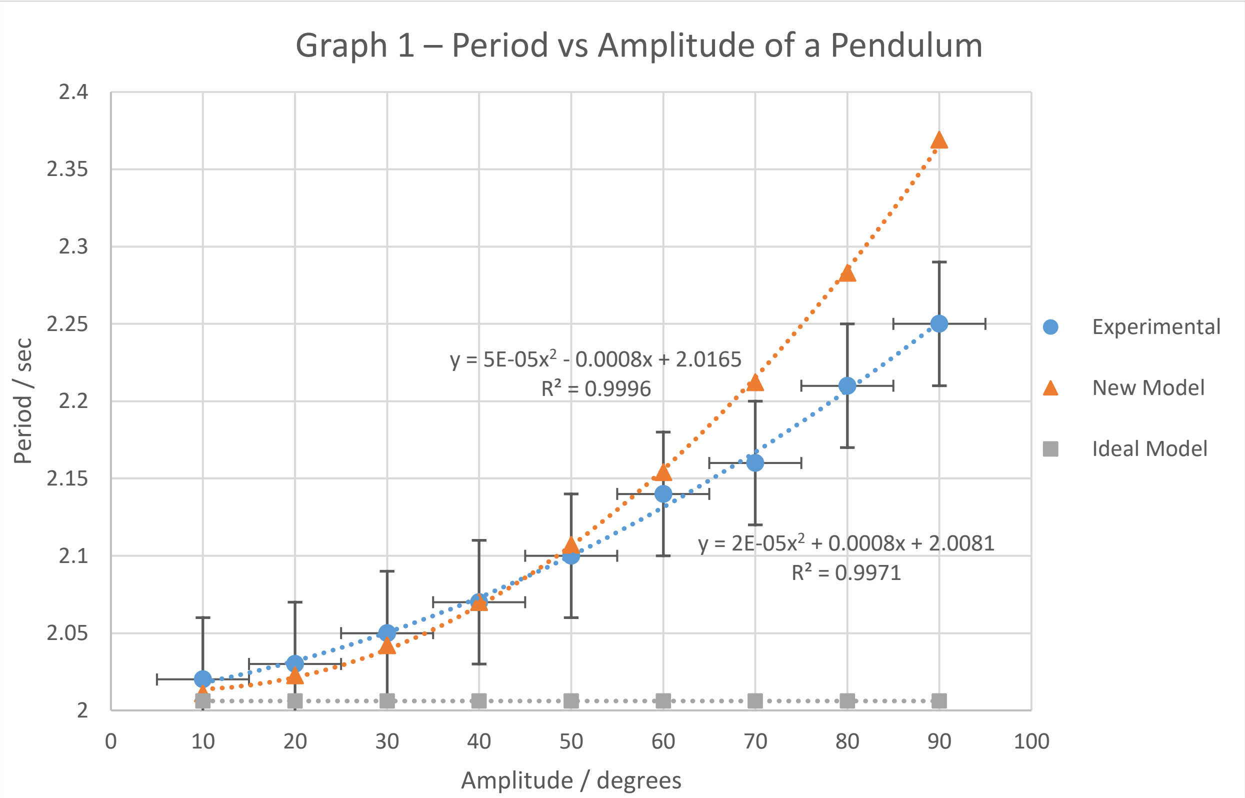
$$T = 4\sqrt{\frac{1}{2 * 9.8}} * \int_0^{10} \frac{1}{\sqrt{cos\theta - \cos(10)}} d\theta = 2.01092\ s$$

The experimental period, and the two calculated periods for each amplitude can be found in Table 2.

Table 2 – Comparison of Experimental and Calculated Periods

Amplitude / degrees Δθ = ±5°	**Experimental Period / sec ΔT = ±0.04 s**	**Simple Model Period / sec**	**Large Amplitude Model Period / sec**
10	2.02	2.006066681	2.01092
20	2.03	2.006066681	2.02248
30	2.05	2.006066681	2.04203
40	2.07	2.006066681	2.06999
50	2.10	2.006066681	2.10701
60	2.14	2.006066681	2.15397
70	2.16	2.006066681	2.2121
80	2.21	2.006066681	2.28305
90	2.25	2.006066681	2.36905

These results are displayed visually in Graph 1 below.



As shown in Graph 1, the new model approximates the experimental values until an amplitude of 60 degrees. When the amplitude is greater than 60, the new model is no longer within the error bars. This is clearly a huge improvement over the ideal/simple model, which is only applicable at much smaller angles (until 20 degrees).

Next, the applicability of the models can be evaluated numerically. Table 3 shows the percentage errors of both models, compared to the experimental values.

Table 3 – Percentage Errors of Calculated Periods

Amplitude / degrees $\Delta\theta = \pm 5°$	**% Error of Simple Model**	**% Error of Large Amplitude Model**
10	0.690	0.450
20	1.18	0.370
30	2.14	0.389
40	3.09	0.000483
50	4.47	0.334
60	6.26	0.653
70	7.13	2.41
80	9.23	3.31
90	10.8	5.29

It can be seen from Table 3 that the percentage errors of the new model are significantly smaller than the simple model.

Furthermore, all of the percentage errors for the new model with amplitudes smaller than or equal to 60 degrees have percentage errors that are smaller than 1%. This proves that the new model is an accurate approximation of the period of a large amplitude pendulum, for amplitudes up to 60 degrees.

In contrast, only the first value of the simple pendulum has a percentage error smaller than 1%, indicating that it is only applicable to amplitudes of up to 10 degrees.

Conclusion and Evaluation

The aim of this experiment was to discover the relationship between the amplitude and the period of a pendulum's oscillation. The oscillations of a pendulum were found to not be isochronous, in contrast to what is taught in the classroom. The period of the pendulum actually varies greatly with relation to the amplitude. That is why the period value calculated with the simple pendulum model is only applicable when the amplitude is no greater than 10 degrees.

A new model was derived by avoiding the use of the trigonometric small-angle approximation. It was hypothesized that this model would be significantly more accurate than the conventional model, especially when the amplitude was relatively large. This hypothesis was confirmed by the data. By using an experimental pendulum, it was found that the new model is accurate (within 1 percent of experimental values) to amplitudes of up to 60 degrees, rather than 10 degrees like the old model. In conclusion, it can be said that avoiding the small-angle approximation is critical when considering large-angle pendulums.

There were a few weaknesses in this investigation that could be improved upon. Firstly, neither air resistance nor friction was accounted for. These two factors were the reason why the oscillations were greatly dampened. When the oscillations were dampened, the amplitude gradually became smaller. And, because there is a positive relationship between the amplitude and period (as found in this investigation), this means that the dampening oscillations also resulted in decreasing periods. Over the 10 oscillation duration, this probably caused the measured mean periods to be smaller than expected for the given amplitudes. Furthermore, it was observed during the experiment that the dampening was more noticeable during the large amplitude trials. Therefore, if that was really the case, then the measured periods might have had an increasing systematic error in the negative direction as the amplitude increased. Furthermore, this could be a viable explanation for why the values calculated by the new model were greater than the experimental values when tested with angles over 60 degrees. If this investigation is repeated in a vacuum chamber, or air resistance and friction is accounted for in some way, then it might be found that the new model is actually accurate for all values of the amplitude.

Two more sources of error were that the string wouldn't be completely taut when oscillating with large amplitudes, and that the oscillation wasn't completely linear, but rather slightly circular. Future studies can avoid these problems by using a physical pendulum composed of a rigid, light rod, and attaching it to a smooth, nearly frictionless joint that only rotates around one axis. This would also allow for the investigation of amplitudes greater than 90 degrees, thereby increasing the scope of the study.

The experiment could be made more precise by using a "Vernier" photogate to detect when the pendulum is at its extreme position and hence measure the period, instead of using a handheld stopwatch. Similarly, a more precise way of measuring the amplitude could have been used, such as using a large protractor. Finally, the pendulum could be allowed to oscillate 20 times instead of 10, to further improve precision. All of these improvements would allow for the random error to be minimized.

Works Cited

Kosinski, Robert J. "A Literature Review on Reaction Time." n.d.

Lima, F. M. S. and P. Arun. "An accurate formula for the period of a simple pendulum oscillating beyond the small angle regime." 2006.